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Decay of long-lived massive closed superstring states: exact results

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Abstract: We find a one-parameter family of long-lived physical string states in type II superstring theory. We compute the decay rate by an exact numerical evaluation of the imaginary part of the one-loop propagator. Remarkably, the lifetime rapidly increases with the mass. We find a power-law dependence of the form $\mathcal{T} \cong \text{const}\, g_s^{-2}\ (\text{mass})^{\alpha}$, where the value of α depends on the parameter characterizing the state. For the most stable state in this family, one has $\alpha \cong 5$. The dominant decay channel of these massive string states is by emission of soft massless particles. The quantum states can be viewed semiclassically as closed strings which cannot break during the classical evolution.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings.

Contents

1. Introduction

A basic feature of Superstring theory is the presence of an infinite tower of massive states, with an exponential degeneracy at high mass level. All these states are expected to be unstable, since they can decay by emission of massive or light particles. An important question is what is the lifetime of massive states in superstring theory. This question was investigated in a number of papers in the past, using different methods [1]–[11]. Nevertheless, there are no conclusive results and the whole picture is far from clear. Could there be long-lived states in string theory? The possible existence of quasi stable states in string theory is an exciting subject, since it could have both theoretical as well as phenomenological implications in Cosmology or accelerator physics in models with large extra dimensions.

How to identify possible long-lived string states among the vast number of states with a given mass?

In section 2 we will give a simple semiclassical argument which allows to find canditates for quasi stable quantum states. The basic idea of the argument is as follows. Highly excited string states have large quantum numbers and can be semiclassically described in terms of solitons. Such solutions are classically stable, unless the string breaks. However, for a theory of closed string only, like type II superstrings, classical breaking is possible only if two points of the string touch during the evolution. Thus the problem of identifying stable quantum states is mapped to the problem of finding classical string solutions whose points do not touch during the evolution. Given a classical solution satisfying this requirement, one can write the corresponding quantum state, and compute the lifetime explicitly by the quantum one-loop calculation. When the classical breaking is not possible, this does not prevent that the original string can decay by emitting *light* particles (actually massless, as we will see), for which a semiclassical description does not apply.

With this line of argument, we have identified a one-parameter family of long-lived string states.

The method used here is a direct numerical evaluation of the formula giving the imaginary part of ΔM^2 at one loop. The method is in fact similar to the one used by Okada and Tsuchiya in [4] for the open string state with maximum angular momentum. In that case, it was found a lifetime $\mathcal{T} = \text{cost } M^{-1}$, implying that the open string becomes more unstable for large masses. We have reproduced this result as a check with our method. However, the states we consider here have no analog in open superstring theory, except for the less stable states of the family which correspond to classically breakable configurations.

In section 3 we explain the computation of $\text{Im}(\Delta M^2)$, which leads us to a formula involving a finite sum which is to be evaluated numerically.In this paper we consider type II superstrings in 10 flat uncompactified dimensions.

In section 4 we discuss the results of the numerical analysis.

In the case of those states whose classical analog is a breakable string we find that the lifetime can increase or decrease with increasing mass, depending on the actual quantum realization; the decay channel into two massive pieces is dominant and the numerical results accurately reproduce what is expected from the classical picture of the string breaking.

In the other cases (of quantum states which classically correspond to unbreakable strings) the by far dominant decay channel is the emission of massless particles, mostly of very low energy. We find that the lifetime grows as a power of the mass, and we compute the spectrum of the emitted massless particles.

In section 5 we make some additional comments, in particular we discuss possible gravitational effects, which were not taken in account before, and argue that they are not expected to play an important role.

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Some details of the calculation are described in appendices. In appendix A, we describe the construction of the quantum superstring states whose decay properties are studied in this paper. In appendix B, we construct the corresponding vertex operator. In appendix C we compute the correlators appearing in the two-point amplitudes in terms of theta-functions of the torus and organize the result in a convenient way. In appendix D we compute the fermion correlators and perform the sum over spin structures. In appendix E we give a simple analytic computation of a decay in a particular channel, obtaining a decay rate which agrees with the numerical results. In appendix F we show the shape of the logarithm of the decay rate divided by $2M^2$ for different values of M for some of the analysed cases.

2. String solitons and quantum states

Quantum states with high quantum numbers can be described in terms of a classical soliton. Examples on how a classical description can describe features of the quantum decay with high accuracy were given in ref. [12] (see also section 3). In this section we construct classical soliton solutions which can be associated with the quantum states whose decay rates will be computed in the next sections. A classical soliton description of a quantum state can give an intuition on the possible decay modes of the quantum state. Here we will consider several examples and identify some quantum states which are expected to be long-lived.

2.1 Quantum Hilbert space

We review here some standard facts of the free string in ten flat spacetime dimensions and set the notation (covariant super-vertices will be discussed in the next section and in the appendices A and B). Define $U = -X_0 + X_9$, $V = X_0 + X_9$, and set $U = 2\sqrt{\alpha'}p_v\tau$. Consider strings which, in addition of moving in the X_9 direction, fluctuate in the planes

$$Z^1 = \frac{X^1 + iX^2}{\sqrt{2}}, \qquad Z^2 = \frac{X^3 + iX^4}{\sqrt{2}}.$$

We shall consider states in the NS-NS sector and omit in the whole section 2 the world-sheet fermions from the formulas (their inclusion is straightforward, but not very relevant for the discussion of this section).

The solution to the equations of motion is

$$V = V_L(\sigma^+) + V_R(\sigma^-), \qquad X^i(\sigma,\tau) = X^i_L(\sigma^+) + X^i_R(\sigma^-), \qquad i = 1,\ldots,4, \tag{2.1}$$

where $\sigma_\pm = \sigma \pm \tau$. In addition, one has to solve the two constraints:

$$\begin{aligned} -\sqrt{\alpha'}p_v\partial_+ V_L &= \partial_+ Z^1_L \partial_+ \bar{Z}^1_L + \partial_+ Z^2_L \partial_+ \bar{Z}^2_L, \\ \sqrt{\alpha'}p_v\partial_- V_R &= \partial_- Z^1_R \partial_- \bar{Z}^1_R + \partial_- Z^2_R \partial_- \bar{Z}^2_R. \end{aligned} \tag{2.2}$$

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We will work in the center of mass frame of the X^i where $p_i = 0$. For the complex coordinates, the Fourier mode expansions is ($\sigma \in [0,\pi)$)

$$
\begin{aligned}
Z_R^1 &= i\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[b_{n-}e^{2in(\sigma-\tau)} - b_{n+}^{\dagger}e^{-2in(\sigma-\tau)}\right] \\
Z_L^1 &= i\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[\tilde{b}_{n-}e^{-2in(\sigma+\tau)} - \tilde{b}_{n+}^{\dagger}e^{2in(\sigma+\tau)}\right] \qquad (2.3) \\
Z_R^2 &= i\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[c_{n-}e^{2in(\sigma-\tau)} - c_{n+}^{\dagger}e^{-2in(\sigma-\tau)}\right] \\
Z_L^2 &= i\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[\tilde{c}_{n-}e^{-2in(\sigma+\tau)} - \tilde{c}_{n+}^{\dagger}e^{2in(\sigma+\tau)}\right]. \qquad (2.4)
\end{aligned}
$$

The operators b, c obey the usual commutation relations:

$$
\begin{aligned}
[b_{n\pm}, b_{m\pm}^{\dagger}] &= \delta_{nm}\,, \qquad [\tilde{b}_{n\pm}, \tilde{b}_{m\pm}^{\dagger}] = \delta_{nm}\,, \\
[c_{n\pm}, c_{m\pm}^{\dagger}] &= \delta_{nm}\,, \qquad [\tilde{c}_{n\pm}, \tilde{c}_{m\pm}^{\dagger}] = \delta_{nm}\,. \qquad (2.5)
\end{aligned}
$$

The mass formula is given by

$$\alpha' M^2 = 2(N_R + N_L) - a\,, \qquad N_R = N_L\,, \qquad (2.6)$$

where the number operators are

$$N_R = \sum_{n=1}^{\infty} n(b_{n+}^{\dagger}b_{n+} + b_{n-}^{\dagger}b_{n-} + c_{n+}^{\dagger}c_{n+} + c_{n-}^{\dagger}c_{n-})\,. \qquad (2.7)$$

The expression for N_L is similar, with the change $\{b \to \tilde{b}, c \to \tilde{c}\}$. The normal ordering constant a is $a = 2$ in the NS-NS sector. In this section it will be ignored since we are interested in states with large $N_R = N_L$.

The angular momentum components $J_{12} = J_{12R} + J_{12L}, J_{34} = J_{34R} + J_{34L}$ in the (X_1, X_2) and (X_3, X_4) planes are

$$J_{12R} = \sum_{n=1}^{\infty}(b_{n+}^{\dagger}b_{n+} - b_{n-}^{\dagger}b_{n-})\,, \qquad J_{12L} = \sum_{n=1}^{\infty}(\tilde{b}_{n+}^{\dagger}\tilde{b}_{n+} - \tilde{b}_{n-}^{\dagger}\tilde{b}_{n-})\,, \qquad (2.8)$$

and similar expressions for J_{34R}, J_{34L}, replacing $\{b \to c\}$. The physical Hilbert space is then constructed (in the light-cone gauge) as usual by applying the creation operators to the vacuum Fock state. The operators $b_{n+}^{\dagger}$, $\tilde{b}_{n+}^{\dagger}$ raise one unit of the spin operator J_{12} in the "up" direction, whereas the operators $b_{n-}^{\dagger}$, $\tilde{b}_{n-}^{\dagger}$ raise one unit of the spin operator J_{12} in the "down" direction, and similarly for J_{34} in the 34 plane.

The classical string solutions have to satisfy the constraints (2.2). A well-known example is the rotating string with maximum angular momentum, where the classical solution is given by

$$
\begin{aligned}
Z_1 &= L\cos(2\sigma)e^{2i\tau}\,, \qquad U = 2\sqrt{\alpha'}p_v\tau\,, \qquad V = 2\sqrt{\alpha'}p_u\tau\,, \qquad (2.9) \\
L^2 &= -2\alpha' p_u p_v = \alpha'^2 M^2\,. \qquad (2.10)
\end{aligned}
$$

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The classical solution (2.9) is obtained by setting

$$-b_{1+}^{cl} = b_{1+}^{\dagger cl} = \frac{iL}{2\sqrt{\alpha'}}\,, \qquad -\tilde{b}_{1+}^{cl} = \tilde{b}_{1+}^{\dagger cl} = \frac{iL}{2\sqrt{\alpha'}}\,, \tag{2.11}$$

and $b_{1-}^{cl} = b_{1-}^{\dagger cl} = 0$, $\tilde{b}_{1-}^{cl} = \tilde{b}_{1-}^{\dagger cl} = 0$. With these classical values for the Fourier coefficients we get

$$M = \frac{L}{\alpha'}\,, \qquad J = \frac{L^2}{2\alpha'}\,, \tag{2.12}$$

which are the correct values of the mass M and spin J for the state of maximum angular momentum.

To have the same values of J_R, J_L, the corresponding quantum state must be of the form

$$|\Phi_{J_{\max}}\rangle = (b_{1+}^{\dagger})^N (\tilde{b}_{1+}^{\dagger})^N |0\rangle\,. \tag{2.13}$$

This has

$$N_R = N_L = N\,.$$

Since, classically, $N_R = L^2/4\alpha'$, we have to set

$$N = \frac{L^2}{4\alpha'}\,.$$

The soliton description applies in the large L limit and $L^2/4\alpha'$ does not need to be an integer. In general the soliton is aproximately described by the quantum state (2.13), with N being the closest integer.

2.2 Quantum string states and solitons fluctuating in two planes

In what follows we only make use of the creation operators $b_{1+}^{\dagger}, c_{1+}^{\dagger}$ and $\tilde{b}_{1+}^{\dagger}$, $\tilde{c}_{1+}^{\dagger}$, so for clarity in the notation we define $b^{\dagger} \equiv b_{1+}^{\dagger}$, etc. (i.e. we omit the subindex $\{1+\}$).

In this paper we shall be interested in the following one-parameter family of quantum states:

$$|\Phi_{k,n}\rangle = \frac{1}{(k+n)!(k-n)!}\,(b^{\dagger})^{k+n}(c^{\dagger})^{k-n}(\tilde{b}^{\dagger})^{k-n}(\tilde{c}^{\dagger})^{k+n}|0\rangle\,, \qquad \langle\Phi_n|\Phi_n\rangle = 1\,. \tag{2.14}$$

It has

$$N_R = N_L = 2k\,, \qquad \alpha' M^2 = 8k\,, \tag{2.15}$$

$$J_{12R} = k+n\,, \qquad J_{12L} = k-n\,, \qquad J_{34R} = k-n\,, \qquad J_{34L} = k+n\,, \tag{2.16}$$

so that $J_{12} = J_{34} = 2k$ (the full superstring state including world-sheet fermion modes is constructed in appendix A).

The soliton solution with the same values of $J_{12R}, J_{12L}, J_{34R}, J_{34L}$ is as follows (here $\alpha' = 1$)

$$\begin{aligned} Z_{1R} &= \sqrt{k+n}\; e^{-2i(\sigma-\tau)}\,, \qquad & Z_{1L} &= \sqrt{k-n}\; e^{2i(\sigma+\tau)}\,, \\ Z_{2R} &= \sqrt{k-n}\; e^{-2i(\sigma-\tau)}\,, \qquad & Z_{2L} &= \sqrt{k+n}\; e^{2i(\sigma+\tau)}\,, \end{aligned} \tag{2.17}$$

or

$$Z_1 = e^{2i\tau}\left(\sqrt{k+n}\; e^{-2i\sigma} + \sqrt{k-n}\; e^{2i\sigma}\right),$$
$$Z_2 = e^{2i\tau}\left(\sqrt{k-n}\; e^{-2i\sigma} + \sqrt{k+n}\; e^{2i\sigma}\right). \qquad (2.18)$$

It describes a rotating ellipse. It is useful to view it in a rotated frame

$$Z'_1 = \frac{Z_1+Z_2}{\sqrt{2}} = \sqrt{2}L_1 e^{2i\tau}\cos(2\sigma)\,,$$
$$Z'_2 = \frac{-Z_1+Z_2}{i\sqrt{2}} = \sqrt{2}L_2 e^{2i\tau}\sin(2\sigma)\,,$$
$$L_1 = \sqrt{k+n}+\sqrt{k-n}\,, \qquad L_2 = \sqrt{k+n}-\sqrt{k-n}\,. \qquad (2.19)$$

Two special cases are $n=0$ and $n=k$.

For $n=k$, one has $L_1 = L_2$ and the classical configuration is a string whose projection in both the $Z_{1,2}$ planes is a circle. This circular string rotates around its center like a wheel, so that the classical distribution of string matter is stationary.

For $n=0$ the solution becomes a straight rotating string. Indeed, the resulting solution

$$n=0: \qquad Z_1 = Z_2 = 2\sqrt{k}e^{2i\tau}\;\cos(2\sigma)\,, \qquad (2.20)$$

is classically equivalent to the string rotating in one-plane, as is clear in the rotated frame $Z'_1 = 2\sqrt{2k}e^{2i\tau}\cos(2\sigma), Z'_2 = 0$ (cf. eq. (2.9)).

However, at quantum level, the state

$$|\Phi_{k,0}\rangle = \frac{1}{(k!)^2}\;(b^\dagger)^k(c^\dagger)^k(\tilde{b}^\dagger)^k(\tilde{c}^\dagger)^k|0\rangle\,, \qquad (2.21)$$

is physically inequivalent from the state

$$\left|\Phi_k^{J_{\max}}\right\rangle = \frac{1}{(2k)!}\;(b'^\dagger)^{2k}(\tilde{b}'^\dagger)^{2k}|0\rangle\,, \qquad (2.22)$$

representing a string rotating in a single plane, with $N_R = N_L = 2k$ and maximal angular momentum, which we call $J_{\max}$ state.

Indeed, the state (2.22) is an eigenstate of $J^2 = \sum_{1\leq i<j\leq 9}(J_{ijR}+J_{ijL})^2$ with eigenvalue $J^2 = 16k^2+28k$ (maximal angular momentum for the given mass). The states (2.14) however are not eigenstates of J^2. In fact

$$J^2|\Phi_{k,n}\rangle = (12k^2-4n^2+28k)|\Phi_{k,n}\rangle + 2(k-n)^2|\Phi_{k,n+1}\rangle + 2(k+n)^2|\Phi_{k,n-1}\rangle \qquad (2.23)$$

In the large k limit, the states $|\Phi_{k,n}\rangle$, $|\Phi_{k,n\pm1}\rangle$ represent essentially the same semiclassical solution (having the same values of E, J_{12}, J_{34}, and approximately the same values of J_{12R}, J_{12L} and J_{34R}, J_{34L}) so effectively $J^2 \cong (12k^2-4n^2+28k)+2(k-n)^2+2(k+n)^2 = 16k^2+28k \cong 16k^2$, which agrees with the value of J^2 of the semiclassical solution.[1]

In the next section we will find that the decay rates of the state $|\Phi_{k,0}\rangle$ (corresponding to $n=0$) share some qualitative features with the decay rates for the state $|\Phi_k^{J_{\max}}\rangle$ (2.22) computed in [11], though the quantitative details of the quantum decay are different.

[1]Note that all the states (2.14) and (2.22) are eigenstates of J_R^2 and J_L^2 with the same eigenvalue $= 4k^2+14k$; in the interactions, however, the conserved operator is J^2.

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2.3 Classically unbreakable closed strings

An important feature of these solutions is that for $n \neq 0$ the closed string cannot break classically in type II superstring theory. This splitting process can only happen if during the evolution there is a time τ_0 where two points of the string get in contact, i.e. there are two values σ_1, σ_2 such that $X^\mu(\sigma_1, \tau_0) = X^\mu(\sigma_2, \tau_0)$. For the straight rotating (folded) closed string (2.9), the breaking can occur at any time. In fact, it was found in [11, 12] that the string can decay into two strings of masses M_1, M_2, where M_2 is a function of M_1. This relation between M_1, M_2 which emerges in the quantum calculation can be understood (and, in fact, accurately described) in terms of the semiclassical process of splitting.

When the classical breaking is not possible, one expects that the decay channel into two large masses M_1, M_2 will be exponentially suppressed (one can interpret this by saying that breaking is possible only by tunnelling effect). The reason is that for large masses M, M_1, M_2 each string in the process should have a soliton semiclassical description, so that the classical approximation is expected to apply, modulo terms which are exponentially small in the masses. The original string can nevertheless decay by emitting *light* particles, for which a semiclassical soliton description does not apply.

This intuition will be confirmed by the quantum calculation in the next section. Moreover, in the cases we consider, we will find that only the rate of *massless* particle emission is not suppressed, with a rate that decreases as the mass increases. The conclusion would be that the lifetime of quantum states associated with general classically unbreakable states should always be very large, since only decay into massless particles is relevant.

It should be noted that there are numerous classical string solutions where the classical breaking is not possible. In particular, one can consider small perturbations of the rotating ellipse solutions,

$$Z^1 = Z^1_{\text{ellipse}} + \epsilon(z_{1R}(\sigma^+) + z_{1L}(\sigma^-))\,, \qquad Z^2 = Z^2_{\text{ellipse}} + \epsilon(z_{2R}(\sigma^+) + z_{2L}(\sigma^-))\,, \quad (2.24)$$

etc., where $Z^{1,2}_{\text{ellipse}}$ is the solution given in (2.18) and $\epsilon \ll 1$. Starting with a given "unbreakable" classical solution one can construct the corresponding quantum state. For example, one may consider the solutions

$$Z_1 = e^{2i\tau}\left(\sqrt{k+n}\ e^{-2i\sigma} + \sqrt{k-n'}\ e^{2i\sigma}\right), \qquad Z_2 = e^{2i\tau}\left(\sqrt{k-n}\ e^{-2i\sigma} + \sqrt{k+n'}\ e^{2i\sigma}\right),$$

which cannot break, indicating that the corresponding quantum states with the same values of J_{12R}, J_{12L} and J_{34R}, J_{34L},

$$|\Phi_{k,n,n'}\rangle = \mathcal{N}\ (b^\dagger)^{k+n}(c^\dagger)^{k-n}(\tilde{b}^\dagger)^{k-n'}(\tilde{c}^\dagger)^{k+n'}|0\rangle\,, \quad (2.25)$$

$$\mathcal{N} = \left[(k+n)!(k-n)!(k+n')!(k-n')!\right]^{-1/2} \quad (2.26)$$

should be long-lived.[2]

[2]The classical solution is related to (2.18) by an $O(4)$ rotation, but the quantum states $|\Phi_{k,n,n'}\rangle$ and $|\Phi_{k,n}\rangle$ are inequivalent.

3. Computation of $\mathrm{Im}(\Delta M^2)$

We consider type II Superstring Theory in ten flat uncompactified dimensions.. The expression for the one-loop mass shift for the state $|\Phi_{k,n}\rangle$, with square mass $M^2_{k,n} = N = 2k-1$ in units $\alpha' = 4$, is (see appendix C):

$$\Delta M^2_{k,n} = c'\ g_s^2 \int \frac{d^2\tau}{\tau_2^3} \int d^2z\ e^{-4N\frac{\pi y^2}{\tau_2}} \left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^{4N} \left(\frac{\pi}{\tau_2}\right)^{2N} \times \\ \times \sum_{m_1,m_2} \left(\frac{\pi}{\tau_2}\right)^{-m_1-m_2} Q(k,n;m_1,m_2)(\partial^2 \log(\theta_1(z|\tau))^{m_1}(\bar\partial^2 \log(\bar\theta_1(z|\tau))^{m_2}\,, \tag{3.1}$$

where the numerical coefficient $Q(k,n;m_1,m_2)$ has been defined in the appendix C, and c' is an overall constant normalization coefficient, which is independent of k and n. It can be seen that this expression is modular invariant.

In order to evaluate the imaginary part of $\Delta M^2_{k,n}$ we expand the holomorphic (and antiholomorphic) factors in powers of $e^{i2\pi\tau}$ and $e^{i2\pi z}$ (respectively of $e^{-i2\pi\bar\tau}$ and $e^{-i2\pi\bar z}$).

It is convenient to define the coefficients $\gamma(m,p,q)$ by the following expansion:

$$(e^{i2\pi z})^N \left(\frac{2\pi\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right)^{2N} \left(\frac{1}{4\pi^2}\partial^2\log(\theta_1(z|\tau)\right)^{m_1} = \sum_{p,q} \gamma(N,m_1;p,q)\ e^{i2p\pi\tau}e^{i2(q-p)\pi z} \tag{3.2}$$

and similarly for the antiholomorphic coefficient $\gamma(N,m_2;\tilde p,\tilde q)$.

We have to evaluate the imaginary part of the integral

$$H \equiv \int \frac{d^2\tau}{\tau_2^{3+2N-m_1-m_2}}\ e^{i2(p-\tilde p)\pi\tau_1 - 2(p+\tilde p)\pi\tau_2} \times \\ \times \int d^2z\ e^{-4N\frac{\pi y^2}{\tau_2} - 2(q-p+\tilde q-\tilde p-2N)\pi y + i2(q-p-\tilde q+\tilde p)\pi x}\,. \tag{3.3}$$

The integrations over τ_1 and over x set $p=\tilde p$ and $q=\tilde q$. Then we use the formula

$$\mathrm{Im}(H) = \mathrm{Im}\int \frac{d\tau_2}{(\pi\tau_2)^{3+2N-m_1-m_2}} e^{-4p\pi\tau_2} \int dy\ e^{-4N\frac{\pi y^2}{\tau_2} - 4(q-p-N)\pi y} \\ = c'' \frac{(N\omega)^{2N-m_1-m_2+3/2}}{\sqrt{N}\Gamma(2N-m_1-m_2+5/2)}\,, \tag{3.4}$$

where c'' is a numerical constant independent of N, p, q, m_1, m_2 and

$$\omega = 1 - 2\left(\frac{p}{N}+\frac{q}{N}\right) + \left(\frac{p}{N}-\frac{q}{N}\right)^2\,,$$

is the typical phase-space function. This can be seen by comparing the integrand in eq. (3.4) with the Schwinger parametrization of a Feynman one-loop diagram for a field theory with vertex $\sim \Phi\ \phi_1\phi_2$, with Φ representing the field of mass M, and $\phi_{1,2}$ the fields of masses $M_{1,2}$. Then one sees that the integral leads to the one-loop correction ΔM^2 due to the process $\Phi \to \phi_1 + \phi_2$ with

$$M_1^2 = q \qquad \text{and} \qquad M_2^2 = p\,.$$

The number $2N - m_1 - m_2 - 2$ is related to the orbital angular momentum carried in the interaction (the $M_{1,2}$ particles having in general intrinsic spins).

The function $\omega = 1 - 2(\frac{M_1^2}{M^2} + \frac{M_2^2}{M^2}) + (\frac{M_1^2}{M^2} - \frac{M_2^2}{M^2})^2$ vanishes at the border of phase space $M_1 + M_2 = M$.

In conclusion, we get for the channel $q = M_1^2, p = M_2^2$

$$\begin{aligned}\mathrm{Im}\Delta M_{k,n}^2(p,q) &= c_0\ g_s^2 \sum_{m_1,m_2} Q(k,n;m_1,m_2)\gamma(m_1,p,q)\gamma(m_2,p,q) \times \\ &\times \frac{N^{2N-m_1-m_2+1}\omega^{2N-m_1-m_2+3/2}}{4^{2N-m_1-m_2}\Gamma(2N-m_1-m_2+5/2)}\,, \end{aligned} \tag{3.5}$$

where c_0 is a numerical constant independent of k, n, p, q, which we conventionally take as $c_0 = 32(2\pi)^3$. We also recall $M^2 = N = 2k - 1$ ($\alpha' = 4$ units).

The decay rate for a given channel into particles of masses M_1 and M_2 is given by

$$R_{k,n}(p,q) = \frac{\mathrm{Im}\Delta M_{k,n}^2(p,q)}{2\sqrt{N}}\,. \tag{3.6}$$

This includes the contributions from all final states with the same masses M_1, M_2.

The total decay rate is $R_{k,n}^{\mathrm{total}} = \sum_{p,q} R_{k,n}(p,q)$ and the lifetime of the Superstring state $|\Phi_{k,n}\rangle$ is

$$\mathcal{T}_{k,n} = \frac{1}{R_{k,n}^{\mathrm{total}}}\,. \tag{3.7}$$

A final comment. Both $Q(k,n;m_1,m_2)$ and $\gamma(m,p,q)$ are integers which become very large in absolute value by increasing k. They have been dealt with by means of computer programs which manipulate integers without approximations. In this sense we have performed an exact computation. The limitation comes from the machine size and we stopped our computation at mass $M^2 = 129$ for the most stable state and $M^2 = 99$ for the others. We will see that this appears to be enough for determining the asymptotic (power-law) formula of the lifetime.

4. Decay rates and lifetimes

We report here the main results for the decay $M \to M_1 + M_2$.

(Conventions: $c_0 = 32(2\pi)^3$, $\alpha' = 4$).

4.1 Case $n \neq 0$

We have considered explicitly the maximal $n = k$ case and the case $n = k/2$ (the latter is expected to illustrate a generic case $0 < n < k$, with n/k not $n/k \ll 1$).

In agreement with the fact that in this case the string does not break classically, the rate in the channels with both $M_1, M_2 \neq 0$ (*massive channels*) is much suppressed (for $n = k$, the "maximally unbreakable" string, it is completely negligible) compared to the channel in which M_1 or M_2 is equal to zero (*radiation channels*), see tables 1 and 2. In particular, from table 1, one observes that the sum of the decay rates for all the channels with both $M_{1,2} \neq 0$ goes very rapidly (exponentially) to zero.

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M^2	$\lvert\Phi^{J_{\max}}\rangle$	$\lvert\Phi_{k,0}\rangle$	$\lvert\Phi_{k,\frac{k}{2}}\rangle$	$\lvert\Phi_{k,k}\rangle$
19	3.450	2.353	0.373	$5 * 10^{-5}$
39	4.713	2.317	0.097	$1.19 * 10^{-8}$
59	5.321	2.183	0.033	$2.26 * 10^{-11}$
79	5.718	2.070	0.014	$1.15 * 10^{-13}$
99	6.013	1.978	0.007	$1.08 * 10^{-15}$
119				$1.6 * 10^{-17}$
129				$2.2 * 10^{-18}$

Table 1: Decay rates: Sum over massive channels $M_{1,2} \neq 0$.

M^2	$\lvert\Phi^{J_{\max}}\rangle$	$\lvert\Phi_{k,0}\rangle$	$\lvert\Phi_{k,\frac{k}{2}}\rangle$	$\lvert\Phi_{k,k}\rangle$
19	7.83	7.02	2.91	0.249
39	6.19	4.31	1.27	0.050
59	5.46	3.32	0.77	0.019
79	5.02	2.78	0.53	0.010
99	4.70	2.44	0.39	0.0056
119				0.0036
129				0.003

Table 2: Decay rates: Sum over radiation channels M_1 or $M_2 = 0$.

M^2	$\lvert\Phi^{J_{\max}}\rangle$	$\lvert\Phi_{k,0}\rangle$	$\lvert\Phi_{k,\frac{k}{2}}\rangle$	$\lvert\Phi_{k,k}\rangle$
19	0.0886	0.107	0.304	4.010
39	0.0917	0.151	0.729	19.830
59	0.0927	0.181	1.246	51.912
79	0.0931	0.206	1.840	103.711
89				137.934
99	0.0933	0.227	2.510	178.155
109				224.689
119				277,834
129				337.877

Table 3: Lifetimes.

Therefore the dominant decay mode is by emission of a massless particle (table 2).

Figure 1 shows the decay rate for $M_1 = 0$ as a function of M_2^2, in the case $n = k = 40$. We see that it is strongly peaked for $M_2^2 = M^2 - l$ with l finite and small with respect to $M^2 = 2k - 1$. Note that this figure gives the spectrum of the emitted massless particle, since its energy is

$$E = \frac{l}{2M} . \tag{4.1}$$

From this we see that the emitted massless particle is soft.

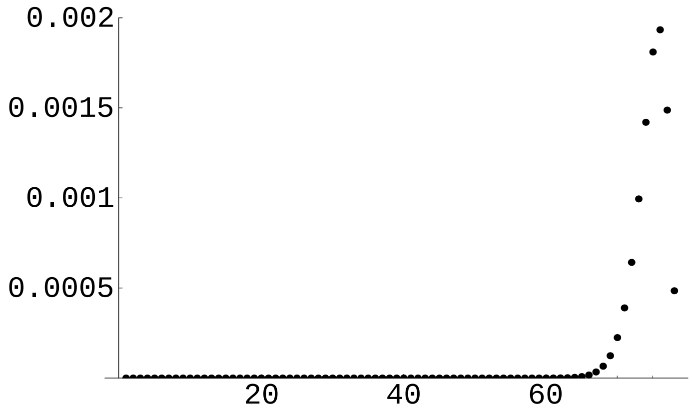

Figure 1: Spectrum of massless particle emission for $n = k = 40, M^2 = 79$.

As for the other decay product, the massive one, we think that it is likely that it is classically similar to the decaying state, although quantum mechanically it can well be different (see also appendix E).

4.2 Case $n = 0$

In this case the rates for the channels $M_{1,2} \neq 0$ (*massive channels*) and for the channels where M_1 or M_2 is zero (*radiation channels*) are comparable.

The massive channels $M_{1,2} \neq 0$ contain the case of classical breaking. In fact we have seen in section 2 that $n = 0$ is the limiting case where the elliptic classical configuration degenerates into a straight line: the classical closed string becomes a folded rotating string which can break at any time.

In ref. [12], by the description of the classical process of the splitting into two closed strings, we obtained the following masses of the decay products:

$$\begin{aligned} M_1(a) &= M \sqrt{a^2 + \frac{\sin^2(\pi a)}{\pi^2}}\,, \\ M_2(a) &= M \sqrt{(1-a)^2 + \frac{\sin^2(\pi a)}{\pi^2}}\,, \end{aligned} \tag{4.2}$$

where $a \in [0, 1]$ is a parameter specifying the splitting point $\sigma_0 = a\pi$. Eq. (4.2) defines parametrically a line $M_1 = M_1(M_2)$.

We expect that the classical breaking configuration with masses (4.2) corresponds to the channel of maximum rate (for $M_{1,2} \neq 0$). Indeed the quantum results for the decay rates are maximal on a curve $M_1 = M_1(M_2)$ in the plane M_1, M_2. As an illustration we plot in figure 2 the logarithm of the decay rate divided by $2M^2$ as a function of M_1^2 for fixed $M_2^2 = 1, 5, 10, 20$ at $M^2 = 99$.

Figure 3 displays the line of maximum value of the decay rate (in the plane M_1/M, M_2/M), showing agreement with the classical curve (4.2). This can be compared with the similar curve obtained in [11] using saddle-point method, which applies at very large M (see also [12]).

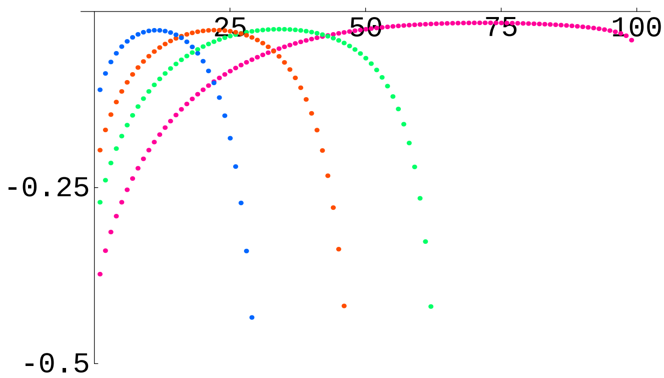

Figure 2: $M^2 = 99$, logarithm of the decay rate divided by $2M^2$ as a function of M_1^2, for $M_2^2 = 1$ (flattest), 5, 10, 20 (increasingly narrower).

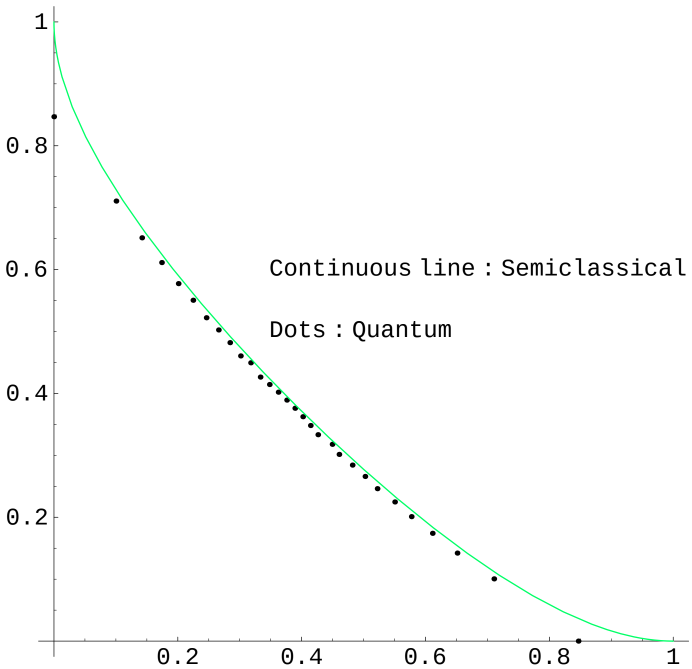

Figure 3: $M^2 = 99$. The continuum line corresponds to the classical $M_1 = M_1(M_2)$ breaking curve (4.2). The dots are the values of M_1, M_2 for which the rate of the quantum decay is maximum.[4]

Near the end points of the classical curve ($M_1 = 0$ or $M_2 = 0$) the process of light or massless particles emission is not described by the classical string breaking. Indeed we see in figure 2 that the rates for $M_2^2 = 1$ are distributed in a rather flat way and it is difficult to see a definite maximum.

The total rate for the sum of all the massive channels where both $M_{1,2} \neq 0$ is seen to decrease with increasing M (table 1, case $|\Phi_{k,0}\rangle$), but less fastly than the sum of the rates of the radiation channels with $M_1 = 0$ or $M_2 = 0$ (table 2, case $|\Phi_{k,0}\rangle$). This is found quantitatively in section 4.3 by a fit of the data with a power like formula.

Another interesting question is to investigate, among the different massive channels with both M_1 and M_2 different from zero, which ones gives the dominant contribution to the decay rate. Figure 2 suggests that light-particle emission could be dominant over other massive channels, and that the behavior for the sum of all massive channels is likely to result from a complex interplay of different individual behaviors, the phase space features playing an important role.

For completeness we have reported in the Tables also the results for the $J_{\max}$ quantum state.

The three-dimensional plot of S_0 — the logarithm of the decay rate divided by $2M^2$ — as a function of M_1, M_2 for the cases $n = 0$ and $J_{\max}$ looks qualitatively the same as the one for the $J_{\max}$ state reported in [11, figure 4]. One interesting question is if the maxima of the curve of figure 3 approach $S_0 = 0$ at large M. This is the expected result, since in the $n = 0$ and $J_{\max}$ cases the semiclassical breaking process should not be exponentially suppressed. In a more detailed view we see that S_0 is rather smaller than zero for small M and shows a number of structures (see appendix F): a peak for $M_1 = M_2$ and shallow maxima towards light values of $M_{1,2}$. By increasing M, those structures tend to disappear. In the case $J_{\max}$ the region of the maxima of S_0 get closer to zero, suggesting that $S_{0\max} \to 0$ for $M \to \infty$. In the case $n = 0$, this is less evident.

In refs. [11] and [12] the study of the $J_{\max}$ state was based on a saddle-point/WKB-type approximation. This gave an accurate determination of the exponential part of the formula for the decay rate, giving $S_{0\max} = 0$ in the $M \to \infty$ limit (precisely for the classical $M_1 = M_1(M_2)$ curve (4.2)) but a less precise indication of the power behavior of the prefactor, and $\alpha = 1$ was reported for the lifetime $\mathcal{T} \sim M^{\alpha}$. We will see in section 4.3 that the precise power behavior of the lifetime as determined by the exact quantum calculation is different, and that there is indeed a long-lived rotating string state $|\Phi_{k,0}\rangle$, although different at quantum level from the $J_{\max}$ state.

4.3 Analysis of the results

To analize in a more detailed way the dominant contribution to the total decay rate, it is convenient to study the different channels separately. The reason is that the decay rate in the channel in which a massless particle is emitted has a different behavior as compared to the channel in which two massive particles are emitted. Considering the cases separately allows for a more accurate convergence to the correct trend for $M \to \infty$. (This separation is relevant only in the $n = 0$ and $J_{\max}$ cases, where the decay rate to two massive particles is significant).

We fit the decay rates with a power-law behaviour, except for the massive channels in the cases $|\Phi_{k,k/2}\rangle$, $|\Phi_{k,k}\rangle$ which go very fast to zero.

To account for subleading corrections, we have fitted the decay rates assuming a dependence on the mass M as follows:

$$f_r^{(1)}(M) = c\ M^{-\alpha} e^{\beta/M} \approx cM^{-\alpha}\left(1 + \frac{\beta}{M}\right) \tag{4.3}$$

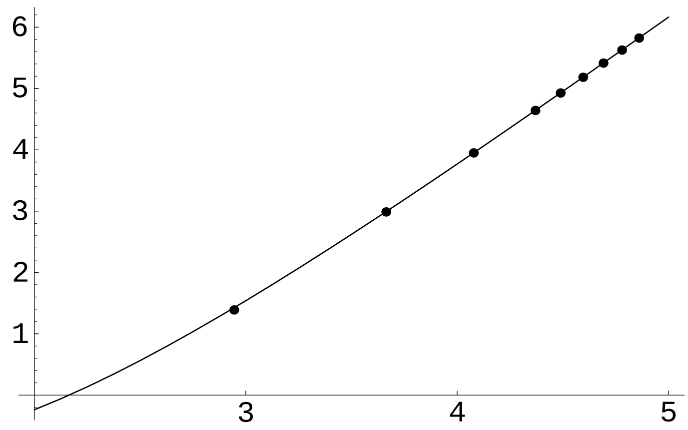

Figure 4: The dots represent the logarithm of the lifetime in the $n = k$ ("circular") case vs. the logarithm of M^2, for the different values of table 3. The continuous line is based on the fit $\log \mathcal{T} = \frac{4.98}{2} \log M^2 + \frac{8.36}{M^2} - 6.35$ (with $v^2 = 4 \cdot 10^{-10}$), see also (4.10).

or

$$f_r^{(2)}(M) = c\ M^{-\alpha} e^{\beta/M^2} \approx cM^{-\alpha}\left(1 + \frac{\beta}{M^2}\right) . \tag{4.4}$$

We considered a favoured fit the one with minimum square deviation. The analysis has been carried through considering the logarithms of the data, and we take as the minimum square deviation $v^2 \equiv \sum_i (\log[r(M_i)] - \log[f_r(M_i)])^2$, where the appropriate $r(M_i)$ can be read from tables 1 and 2.

a) In the $J_{\max}$ case (rotating string in one plane, maximal angular momentum) the decay is dominated by the sum of massive channels, containing the classical breaking. This is most favourably fitted by

$$\Gamma_{\text{massive}} = c\, g_s^2\ M^{0.25} e^{\frac{\beta}{M^2}} . \tag{4.5}$$

with $v^2 = 2 \cdot 10^{-6}$ (or $\Gamma_{\text{massive}} = c\ g_s^2\ M^{-0.14} e^{\frac{\beta}{M}}$, $v^2 = 2 \cdot 10^{-5}$).

We get two comparable fits for the decay rate for the total of radiation channels, where at least one of the decay product is massless:

$$\begin{aligned} \Gamma_0 &= c\ g_s^2\ M^{-0.54} e^{\frac{\beta}{M^2}} \qquad \text{or} \\ \Gamma_0 &= c\ g_s^2\ M^{-0.46} e^{\frac{\beta}{M}} \end{aligned} \tag{4.6}$$

with $v^2 = 8 \cdot 10^{-8}$ in both cases.

b) In the case $n = 0$ the sum of the massive channels (which contains the classical breaking) dominates the decay, with a fit

$$\Gamma_{\text{massive}} = c\ g_s^2\ M^{-0.58} e^{\frac{\beta}{M^2}} , \tag{4.7}$$

with $v^2 = 2 \cdot 10^{-6}$ (or $\Gamma_{\text{massive}} = c\ g_s^2\ M^{-0.90} e^{\frac{\beta}{M}}$, $v^2 = 2 \cdot 10^{-5}$).

The decay rate for the total of radiation channels is fitted by:

$$\Gamma_0 = c\ g_s^2\ M^{-0.96} e^{\frac{\beta}{M}} \tag{4.8}$$

with $v^2 = 7 \cdot 10^{-8}$ (or $\Gamma_0 = c\ g_s^2\ M^{-1.10} e^{\frac{\beta}{M^2}}$, $v^2 = 9 \cdot 10^{-7}$).

c) For the case $n = k/2$, as already mentioned, the sum of the radiation channels, in which M_1 or M_2 is equal to zero, is the by far dominant contribution, with a fit

$$\Gamma_0 = c\, g_s^2\; M^{-3.02} e^{\frac{\beta}{M}}\,, \tag{4.9}$$

with $v^2 = 4 \cdot 10^{-5}$ ($\Gamma_0 = g_s^2 \cdot c \cdot M^{-2.74} e^{\frac{\beta}{M^2}}$ gives $v^2 = 7 \cdot 10^{-5}$).

d) For the case $n = k$ the sum over the radiation channels utterly predominates (the massive channels contribution is absolutely negligible). In this case we have more data for the lifetime given in table 3. Since we are interested in the asymptotic behavior, for the fit we only use the largest values of $M^2 = 79, 89, 99, 109, 119, 129$. We obtain

$$\Gamma_0 = c\, g_s^2\; M^{-4.98} e^{\frac{\beta}{M^2}} \tag{4.10}$$

with $v^2 = 6 \cdot 10^{-10}$ ($\Gamma_0 = c\, g_s^2\; M^{-5.1} e^{\frac{\beta}{M}}$ gives $v^2 = 8 \cdot 10^{-10}$).

Finally, from the above analysis, we compute the lifetime behavior with the mass M, that is the one given by the predominant decay channels only, by:

$$\mathcal{T} = 1/\Gamma(\text{dominant})\,. \tag{4.11}$$

This gives the lifetime behavior for our states in the limit $M \to \infty$. Notice that in the cases $J_{\max}$ and $n = 0$ a naive fit to the total lifetime given in table 3 would not follow this result. In fact, the trend of the massive decay rate, dominating over the rest, is disguised by the fact that the sum of the radiation channels is of comparable magnitude.

In conclusion our results indicate, in the large M limit, for the various cases, a power-like behaviour for the lifetime as:

$$\mathcal{T} = \frac{c}{g_s^2}\; M^{\alpha}\,. \tag{4.12}$$

Cases	α
$J_{\max}$	-0.25
$n = 0$	0.58
$n = k/2$	3.02
$n = k$	4.98

Table 4: Lifetimes for the different cases: $\mathcal{T} \sim M^{\alpha}$. See the discussion in the text for the uncertainty on α.

In table 4 we summarize the values found for α in the various cases. The uncertainty on α can be estimated by comparing different fits, as it has been reported above. For the most interesting "circular" case $n = k$, the value of $\alpha = 4.98$ has a very small uncertainty, as it can also be seen from figure 4. Also the value of $\alpha = 3.0$ for the intermediate case $n = k/2$ has a small uncertainty, although in this case the accuracy in the determination of α is lower than in the $n = k$ case, because the numerical results for $n = k/2$ are so far available only up to $M^2 = 99$. For the breakable cases, $n = 0$ and $J_{\max}$, the values of α given in the table 4 have a larger uncertainty, which is more difficult to estimate, because, as we said, there are competing decay channels of the same magnitude but different behavior (and, moreover, neither in those cases we could obtain numerical results beyond $M^2 = 99$).

In appendix E we report a computation with the operator formalism giving the behavior with M of the decay rate for the channel:

$$\text{state}\{n = k\} \to \text{graviton} \;+\; \text{state}\{n = k - l\}\,. \tag{4.13}$$

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We obtain a lifetime for this mode $\mathcal{T} \cong c(l)\ M^5$ for any l small with respect to k (that is to M^2). This is very near the result $\alpha = 4.98$, and suggests that the channels $(M_1^2 = 0) + (M_2^2 = M^2 - l)$ dominate the decay (see figure 1). Note that, a priori, the power law behavior of the decay rate for this particular channel (4.13) needs not be the same as the formula for the full decay rate, since there are many other states with $M_2^2 = M^2 - l$.

5. Discussion

To summarize, the exact evaluation of the one-loop contribution to $\mathrm{Im}(\Delta M^2)$ shows that there are massive states in type II superstring theory which are almost stable, with a lifetime growing with the mass as fast as $\mathcal{T} = \text{cost}\ g_s^{-2}\ M^5$.

We have seen that these states can decay only by masless emission (other channels being exponentially suppressed), in agreement with the interpretation that they cannot break classically. We repeat that these are superstring type II states. Indeed they do not exist in type I superstring theory, due to the fact that the type I superstring is unoriented.

A natural question is how the decay rates computed here are corrected by higher loops. One particular sector of higher loop corrections are gravitational effects, which are expected to be of order $g_s^2 M$. Therefore they should not affect the formula for the lifetime $\mathcal{T} = \text{cost}\ g_s^{-2}\ M^5$ as long as $M \ll 1/g_s^2$ (in units of α'). This indicates that the lifetime can be as large as $\mathcal{T} = \text{cost}\ g_s^{-12}$!

For states with masses larger than $1/g_s^2$, gravitational effects can no longer be ignored. However, note that none of these states can become black holes at larger masses. The reason is that the spatial extension of these states grows linearly with the mass, i.e. $L \sim M/\alpha'$ (see section 2), whereas in ten dimensions the Schwarzschild radius grows with the mass as $R_{\text{sch}} \sim M^{1/7}$, i.e. for large masses it is always the case that the size L is much larger than R_{sch}. As M is increased, the gravitational field near a segment of the string becomes strong, and the question is what happens to the string configuration at larger couplings. In particular, the most stable state of the family, the $n = k$ state representing a rotating circular string, seems a very robust string configuration and it is plausible that it may survive in the strong coupling regime. This state (or a mixture of states associated with small fluctuations (2.24) near this state) is a natural candidate for becoming a black ring [13] in the strong coupling regime. The fact that this rotating $n = k$ string solution exists in $D = 5$ but not in $D = 4$ is consistent with the fact that there is no black ring in four dimensions. It would be interesting to study the correspondence principle [14] for this case.

A question of interest is how many states — among the exponential number of states existing at each mass level — are long-lived. This is obviously a difficult question to answer in general, without an explicit computation case by case. Nevertheless, one can try to estimate the number of classically unbreakable states, since for such states the only relevant decay channel is by emission of light particles. Like in the examples discussed in this paper, they are expected to have a longer lifetime.

It should be noted that the higher is the dimension, the less is probable that two points of the string get in contact (e.g. viewing the average string state as a random walk process). This suggests that in ten dimensions, a type II superstring state has little chance to break.

Therefore, for most states the decay would dominantly occur through energy leakage by radiation of massless (or light) modes (rather than decay into two very massive particles, which requires breaking of the string).

We have found that in the classically breakable $n = 0$ case the lifetime grows as $\mathcal{T} \sim g_s^{-2}\ M^{0.58}$, and that this is determined by the asymptotically dominant channel of string breaking. This would suggest that a *generic* closed string state should also become more stable for larger masses. Indeed, in the $n = 0$ case, the closed string is folded with all points in contact at all times, so it can break at any time. Instead, a generic closed string solution can only break at a discrete set of specific times and at the specific place where two points touch. For this reason, the decay rate into two massive particles of a generic state should be lower than in the $n = 0$ case. Of course, it is possible that for some states the decay channel by massless particle emission is more relevant, with the effect of reducing the lifetime as compared to the $n = 0$ case.

It seems unlikely that the family of states (2.14) we have considered here constitutes the only states that survive the large mass limit (in particular, we expect that the states of the larger family (2.25) are also long-lived). Long-lived states are very visible among the states produced in a high energy collision, and it is plausible that there is a vast sector of long-lived states in type II superstring theory which could be produced, i.e. a vast sector of states whose lifetime increases with the mass. This is an exciting prospect which deserves further investigation.

Acknowledgments

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A. Quantum state construction

As specified in section 2 we are interested in one particular family of superstring states. In eq. (2.14) we introduced only the bosonic part of our states. Here we construct the full Superstring state.

Define Z^1 and Z^2 as in section 2, with the expansions (2.3), (2.4). The bosonic contribution to the Superstring state is as in eq. (2.14), which we repeat for convenience:

$$|\Phi_{k,n}\rangle = a_n\ (b^\dagger)^{k+n}(c^\dagger)^{k-n}(\tilde{b}^\dagger)^{k-n}(\tilde{c}^\dagger)^{k+n}|0\rangle\,, \tag{A.1}$$

where a_n is a normalization factor:

$$a_n = \frac{1}{(k+n)!(k-n)!}\,. \tag{A.2}$$

Now we proceed to the construction of the superstate. We look for a state with the same values of energy and angular momentum components. The superstate satisfying these requirements is given by:

$$|\Phi_{k,n}\rangle = \mathcal{N} a_n\ |\phi_n^R\rangle|\phi_n^L\rangle \tag{A.3}$$

with

$$|\phi_n^R\rangle = \left((k+n) \left(\psi^{z_1}_{-\frac{1}{2}}\right)^\dagger c_1^\dagger + (k-n) \left(\psi^{z_2}_{-\frac{1}{2}}\right)^\dagger b_1^\dagger \right) \left(b_1^\dagger\right)^{k+n-1} \left(c_1^\dagger\right)^{k-n-1} |0,p\rangle \,, \tag{A.4}$$

where we have written only the right-moving part (the left-moving is obtained by substituting all the oscillators with tilded operators and changing $n \to -n$). It is obtained from (A.1) by applying the mode

$$G_{1/2} = \sum_{n\epsilon\mathbb{Z}} \alpha_n^\mu \; \psi_{\mu \; \frac{1}{2}-n}$$

of the superconformal current (the analog must be done in the left-moving sector). Here we have used real coordinate modes in order to simplify the notation.

The state (A.4) satisfies all the physical constraints, namely (we write only those of the right-moving sector)

$$\begin{aligned} G_r|\Phi_{k,n}\rangle &= 0\,, \qquad r \geq \frac{1}{2}\,, \\ L_n|\Phi_{k,n}\rangle &= 0\,, \qquad n \geq 1\,, \\ \left(L_0 - \frac{1}{2}\right)|\Phi_{k,n}\rangle &= 0\,, \end{aligned} \tag{A.5}$$

where L_n, G_r are the super-Virasoro algebra generators and we are in the Neveu-Schwarz sector (the zero-point energy is equal to 1/2).

The normalization constant in front of the state can be easily computed by requiring $\langle\Phi_{k,n}|\Phi_{k,n}\rangle = 1$, giving:

$$\mathcal{N} = \frac{1}{2k}\,, \tag{A.6}$$

i.e. an additional factor $1/2k$ with respect to the bosonic state (A.1).

Finally, one can check that the mass-squared operator M^2 acting on this state gives

$$\alpha' M^2 = 4(2k-1)\,. \tag{A.7}$$

B. Vertex operators

Here we construct the vertex operators corresponding to the state we have introduced in the preceding section. From now on we will consider explicitly only the right-moving (holomorphic) part of the complete vertex. The construction of the left-moving (anti-holomorphic) part is similar.

The vertex operator for a superstring state can be constructed in different pictures. Following the definitions of [15], we shall use the "integrated" form of the vertex operator.

Given an operator W of conformal dimension $h_W = 1/2$, a vertex operator V of conformal dimension $h = 1$ can be obtained by writing

$$V = G_{-\frac{1}{2}} \cdot W\,, \tag{B.1}$$

where "$A \cdot B$ " indicates the usual commutator (or anticommutator) of the operators A, B.[5]

[5] An equivalent way to construct a vertex operator is to define $V = G_r \cdot W$ by requiring V to be independent of r. We have also pursued this way of computing V finding that indeed it leads to the same final expression.

As operator W we take

$$W_{k,n} = ((k+n)\psi^{z_1}\partial Z_2 + (k-n)\psi^{z_2}\partial Z_1)\,\partial {Z_1}^{k+n-1}\partial {Z_2}^{k-n-1}\,(\text{rightmovers})e^{ip\cdot X}\,, \quad \text{(B.2)}$$

which has conformal dimension $h_W = 1/2$ provided $p^2 = 2k-1$.

Acting on this operator with the superconformal charge produces a certain number of terms. We are interested only in the terms which will give a non-vanishing contribution to the one-loop amplitude. As shown in appendix D, only a few terms with a specific combination of fermionic operators give a non-zero contribution in the two-point function on the torus.

Considering therefore only these terms, our final vertex operator is

$$V_{k,n} = \text{Norm} V^L V^R e^{ip\cdot X}\,,$$

with

$$\begin{aligned} V^R &= (\psi^{z_1}\partial\psi^{z_2} + \psi^{z_2}\partial\psi^{z_1})(k^2-n^2)(\partial Z_1)^{k+n-1}(\partial Z_2)^{k-n-1} + \\ &\quad + (k+n)(k+n-1)\psi^{z_1}\partial\psi^{z_1}(\partial Z_1)^{k+n-2}(\partial Z_2)^{k-n} + \\ &\quad + (k-n)(k-n-1)\psi^{z_2}\partial\psi^{z_2}(\partial Z_1)^{k+n}(\partial Z_2)^{k-n-2}\,. \end{aligned} \quad \text{(B.3)}$$

The *Left* part of the vertex $V^L_{k,n}$ is the same with $\partial \to \bar\partial$ and $\psi \to \tilde\psi$.

We recall that the normalization constant is:

$$\text{Norm} = \frac{1}{2\text{k}(\text{k}+\text{n})!(\text{k}-\text{n})!}\left(\sqrt{\frac{2}{\alpha'}}\right)^{4\text{k}-4}\,. \quad \text{(B.4)}$$

In what follows we choose units where $\alpha' = 4$.

C. Computation of the amplitude

The correlator $\langle \bar V_{k,n} V_{k,n}\rangle$ on the torus is (after summing over spin structures, see appendix D):

$$\langle \bar V_{k,n}(z) V_{k,n}(0)\rangle = \frac{1}{(2k(k+n)!(k-n)!)^2}\left\langle \mathbf{O}^R_{k,n}(z)\mathbf{O}^L_{k,n}(z)\right\rangle \langle e^{-ipX(z)}e^{ipX(0)}\rangle\,, \quad \text{(C.1)}$$

where

$$\begin{aligned} \mathbf{O}^{R,L}_{k,n}(z) &= 4(k^2-n^2)^2 O^{R,L}_\alpha(z,0) + (k+n)^2(k+n-1)^2 O^{R,L}_\beta(z,0) + \\ &\quad + (k-n)^2(k-n-1)^2 O^{R,L}_\gamma(z,0)\,, \end{aligned} \quad \text{(C.2)}$$

and

$$\begin{aligned} O^R_\alpha(z,0) &= (\partial\bar Z_1)^{k+n-1}(z)(\partial\bar Z_2)^{k-n-1}(z)(\partial Z_1)^{k+n-1}(0)(\partial Z_2)^{k-n-1}(0)\,, \\ O^R_\beta(z,0) &= (\partial\bar Z_1)^{k+n-2}(z)(\partial\bar Z_2)^{k-n}(z)(\partial Z_1)^{k+n-2}(0)(\partial Z_2)^{k-n}(0) \\ O^R_\gamma(z,0) &= (\partial\bar Z_1)^{k+n}(z)(\partial\bar Z_2)^{k-n-2}(z)(\partial Z_1)^{k+n}(0)(\partial Z_2)^{k-n-2}(0)\,, \end{aligned}$$

and the "Left" Operators O^L are the same as the corresponding O^R with $\partial \to \bar\partial$.

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We see that we have 9 possible contractions $\langle O^R O^L\rangle$. The basic correlators are

$$\begin{aligned}
\langle \partial \bar{Z}_i(z)\partial Z_j(0)\rangle &= -\delta_{ij}\ 2\left(\partial^2 \log\theta_1(z|\tau)+\frac{\pi}{\tau_2}\right),\\
\langle \bar\partial \bar{Z}_i(z)\bar\partial Z_j(0)\rangle &= -\delta_{ij}\ 2\left(\bar\partial^2 \log\bar\theta_1(z|\tau)+\frac{\pi}{\tau_2}\right),\\
\langle \bar\partial \bar{Z}_i(z)\partial Z_j(0)\rangle &= \langle \partial \bar{Z}_i(z)\bar\partial Z_j(0)\rangle = \delta_{ij}2\ \frac{\pi}{\tau_2},\\
\langle e^{-ipX(z)}e^{ipX(0)}\rangle &= e^{-4(2k-1)\frac{\pi y^2}{\tau_2}}\ \left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^{4(2k-1)}
\end{aligned}\tag{C.3}$$

(here $y=\mathrm{Im}(z)$). The generic term $\langle O^R O^L\rangle$ gives (with the appropriate $m,n,\bar m,\bar n$):

$$\begin{aligned}
&\langle\ (\partial\bar Z_i)^m(z)(\partial Z_i)^n(0)(\bar\partial\bar Z_i)^{\bar m}(z)(\bar\partial Z_i)^{\bar n}(0)\ \rangle =\\
&= \sum_j \frac{m!n!\bar m!\bar n!}{j!\bar j!(m-j)!(\bar m-\bar j)!}\langle\partial\bar Z_i(z)\partial Z_i(0)\rangle^j\langle\bar\partial\bar Z_i(z)\bar\partial Z_i(0)\rangle^{\bar j}\times\\
&\quad\times\langle\partial\bar Z_i(z)\bar\partial Z_j(0)\rangle^{m-j}\langle\bar\partial\bar Z_i(z)\partial Z_j(0)\rangle^{\bar m-\bar j}\,.
\end{aligned}\tag{C.4}$$

Note that in every term the relations $m-j=\bar n-\bar j$, $\bar m-\bar j=n-j$ hold, and thus $\bar j=j-(m-\bar n)=j-(n-\bar m)$. We finally obtain:

$$\begin{aligned}
\langle\bar V_{k,n}(z)V_{k,n}(0)\rangle &= \frac{(k+n)!^2(k-n)!^2}{4k^2}\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^{4(2k-1)}e^{\frac{-4(2k-1)\pi y^2}{\tau_2}}\times\\
&\quad\times\sum_{j,l}c(j,l;k,n)\left(\frac{\pi}{\tau_2}\right)^{2(2k-1-j-l)-2}\left|\partial^2\log\theta_1(z|\tau)+\frac{\tau_2}{\pi}\right|^{2(j+l)}.
\end{aligned}$$

The sum runs over $2n-2\le j\le k+n, 0\le l\le k-n$,

$$c(j,l;k,n)=\frac{\sum_{i=1}^9 S_i(j,l;k,n)}{j!l!(k+n-j)!^2(k-n-l)!^2(j-2n+2)!(l+2n+2)!}\tag{C.5}$$

and

$$\begin{aligned}
\sum_{i=1}^3 S_i(j,l;k,n) &= (k+n-j)^2(k-n-l)^2A(j,l;k,n)\,,\\
\sum_{i=4}^6 S_i(j,l;k,n) &= (k+n-j)^2(k+n-j-1)^2B(j,l;k,n)\,,\\
\sum_{i=7}^9 S_i(j,l;k,n) &= \sum_{i=4}^6 S_i(l,j;k,-n)\,,
\end{aligned}$$

with

$$\begin{aligned}
A(j,l;k,n) &= 4(j-2n+2)(l+2n+2)(l+2n+1)(l+2n)+\\
&\quad+4(j-2n+2)(j-2n+1)(j-2n)(l+2n+2)+\\
&\quad+16(j-2n+2)(j-2n+1)(l+2n+2)(l+2n+1)\,,
\end{aligned}\tag{C.6}$$

$$\begin{aligned}
B(j,l;k,n) &= 4(j-2n+2)(l+2n+2)(l+2n+1)(l+2n)+\\
&\quad+(l+2n+2)(l+2n+1)(l+2n)(l+2n-1)+\\
&\quad+(j-2n+2)(j-2n+1)(l+2n+2)(l+2n+1)\,.
\end{aligned}\tag{C.7}$$

Note that for the particular case $n = k$ the above expressions simplify, since the only nonvanishing contribution occurs for $j = 2k-2, l = 0$ and $c(2k-2, 0; k, k) = 1/(2k-2)!^2$.

Expanding the holomorphic and antiholomorphic binomial factors we obtain

$$\begin{aligned}\langle \bar{V}_{k,n}(z)V_{k,n}(0)\rangle \; &= e^{-4(2k-1)\frac{\pi y^2}{\tau_2}} \left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^{4(2k-1)} \left(\frac{\pi}{\tau_2}\right)^{2(2k-1)-2} \times \\ &\times \sum_{m_1,m_2} \left(\frac{\pi}{\tau_2}\right)^{-m_1-m_2} Q(k,n;m_1,m_2)(\partial^2 \log(\theta_1(z|\tau))^{m_1}(\bar{\partial}^2 \log(\theta_1(z|\tau))^{m_2} ,\end{aligned}$$

where

$$Q(k,n;m_1,m_2) = \frac{(k+n)!^2(k-n)!^2}{4k^2}\sum_{j,l}\frac{c(j,l;k,n)(j+l)!^2}{m_1!m_2!(j+l-m_1)!(j+l-m_2)!} . \tag{C.8}$$

In the particular case $n = k$ we get

$$Q(k,k;m_1,m_2) = \frac{(2k-1)!^2}{m_1!m_2!(2k-2-m_1)!(2k-2-m_2)!} . \tag{C.9}$$

The computation for this $n = k$ case can be done directly from the vertex

$$V_{k,k} = \frac{1}{2^{2k-2}}\frac{(2k-1)^2}{(2k-1)!^2}\psi_{z_1}\partial\psi_{z_1}(\partial Z_1)^{2k-2}\tilde{\psi}_{z_2}\bar{\partial}\tilde{\psi}_{z_2}(\bar{\partial}Z_2)^{2k-2}e^{ipX} , \tag{C.10}$$

noticing that in this case

$$\langle \bar{V}_{k,k}(z)V_{k,k}(0)\rangle = (2k-1)^2\langle e^{-ipX(z)}e^{ipX(0)}\rangle \left|\partial^2 \log\theta_1(z|\tau) + \frac{\pi}{\tau_2}\right|^{2(2k-2)} . \tag{C.11}$$

Another state of interest is the state of maximal angular momentum with $M^2 = 2k-1$ which rotates in one plane only. In this case the superstate is (cf. eq. (2.22), for the bosonic version)

$$|\Phi_k^{J\max}\rangle = \frac{1}{(2k-1)!}{\psi_1}^{\dagger}(b^{\dagger})^{2k-1}\tilde{\psi}_1^{\dagger}(\tilde{b}^{\dagger})^{2k-1}|0\rangle . \tag{C.12}$$

The corresponding vertex is given by

$$V_k^{J\max} = \frac{1}{2^{2k-2}}\frac{(2k-1)^2}{(2k-1)!}\psi_{z_1}\partial\psi_{z_1}(\partial Z_1)^{2k-2}\tilde{\psi}_{z_1}\bar{\partial}\tilde{\psi}_{z_1}(\bar{\partial}Z_1)^{2k-2}e^{ipX} , \tag{C.13}$$

and

$$\begin{aligned}\langle \bar{V}_k^{J\max}(z)V_k^{J\max}(0)\rangle \; &= \langle e^{-ipX(z)}e^{ipX(0)}\rangle \times \\ &\times \sum_j \frac{(2k-1)!^2}{j!^2(2k-2-j)!^2}\left|\partial^2\log\theta_1(z|\tau) + \frac{\pi}{\tau_2}\right|^{2j}\left(\frac{\pi}{\tau_2}\right)^{2(2k-1)-2j-2} .\end{aligned}$$

One obtains

$$Q^{J\max}(k,m_1,m_2) = \frac{(2k-1)!^2(2(2k-1)-m_1-m_2)!}{m_1!m_2!(2k-2-m_1)!^2(2k-2-m_2)!^2} . \tag{C.14}$$

In all cases ΔM^2 is finally obtained by the integral:

$$\Delta M^2 = \int \frac{d^2\tau}{\tau_2^5} \int d^2z \ \langle \bar{V}(z) V(0) \rangle \tag{C.15}$$

by inserting the appropriate vertex.

We will use convenient formulas for the expression involved in the computation of the amplitude ([4, 15])

$$\begin{aligned} 2\pi i \left(e^{2\pi i z}\right)^{\frac{1}{2}} \frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)} &= - \left(\prod_{n=1}^{\infty} [1-(x_1x_2)^n]^3 \right)^{-1} \times \\ &\quad \times \left(\sum_{n=1}^{\infty} (-1)^{n-1} x_1^{n(n-1)/2} x_2^{(n-1)(n-2)/2} (x_2^{2n-1} - 1) \right) \\ \frac{1}{4\pi^2} \partial_z^2 (\log \theta_1(z,\tau)) &= \sum_{n=1}^{\infty} n x_2^n + \sum_{n,l=1}^{\infty} (x_1x_2)^{nl} (x_2^n + x_2^{-n}) \end{aligned} \tag{C.16}$$

with $x_1 = e^{2\pi i(\tau - z)}$, $x_2 = e^{2\pi i z}$.

D. Sum over spin structures

In the NS-R formulation of type II superstring theory, the product of functional determinants of bosonic, fermionic coordinates and ghosts for a definite spin structure s is (Right sector)

$$Z_s = \left(\frac{\theta_s(0)}{\theta_1'(0)} \right)^4 .$$

The other ingredient which depends on the spin strucure s is the fermionic correlator:

$$\langle \Psi_\mu(z) \Psi_\nu(0) \rangle_s \langle \Psi_\lambda(w) \Psi_\rho(0) \rangle_s = g_{\mu\nu} g_{\lambda\rho} \frac{\theta_s(z)\theta_s(w)(\theta_1'(0))^2}{\theta_1(z)\theta_1(w)(\theta_s(0))^2} .$$

Summing over spin structures with the GSO phases η_s, we get

$$\sum_s \eta_s Z_s \langle \Psi_\mu(z) \Psi_\nu(0) \rangle_s \langle \Psi_\lambda(w) \Psi_\rho(0) \rangle_s = g_{\mu\nu} g_{\lambda\rho} \frac{\left(\theta_1(\frac{z+w}{2})\right)^2 \left(\theta_1(\frac{z-w}{2})\right)^2}{\theta_1(z)\theta_1(w)\left(\theta_1'(0)\right)^2} . \tag{D.1}$$

Since $\theta_1(0) = 0$ we obtain a vanishing result for $z \to w$.

Now consider correlators involving $\partial\psi$. Making first the derivative in z and w we get a non-vanishing result for $z \to w$:

$$\lim_{z \to w} \sum_s \eta_s Z_s \langle \partial \Psi_\mu(z) \Psi_\nu(0) \rangle_s \langle \Psi_\lambda(w) \partial \Psi_\rho(0) \rangle_s = g_{\mu\nu} g_{\lambda\rho} \frac{\left(\theta_1(z)\right)^2 \left(\theta_1'(0)\right)^2}{\left(\theta_1(z)\right)^2 \left(\theta_1'(0)\right)^2} = g_{\mu\nu} g_{\lambda\rho} . \tag{D.2}$$

This is the basic correlator used in appendix C in order to obtain eq. (C.1).

E. Analytic determination of a decay rate

We study here the decay of the state $n = k$ in a particular channel where one of the decay products is massless, by using the operator formalism.

Let us for short use the following notation for the state $n = k$ (in this notation $N = 2k - 1$):

$$|\Phi_N\rangle \equiv \frac{1}{N!}\,\psi_1^\dagger (b^\dagger)^N \tilde{\psi}_2^\dagger (\tilde{c}^\dagger)^N |0\rangle\,. \tag{E.1}$$

We will compute the decay-rate of the process in which the state $n = k$ decays into a massless particle ("graviton") and another massive state of the same kind:

$$|\Phi_N\rangle \to |\text{graviton}\rangle + |\Phi_{N-l}\rangle\,. \tag{E.2}$$

We are interested in the behavior of the decay rate for $N \gg 1$ and l finite.

Thus we consider the matrix element

$$A = g_s A_R A_L\,, \tag{E.3}$$

where

$$A_R = \langle 0| \frac{b^{N-l}}{\sqrt{(N-l)!}} \psi_1\ \epsilon\cdot b \exp\left(-\sqrt{2}p\cdot b\right)\ \frac{\psi_1^\dagger b^{\dagger N}}{\sqrt{N!}}|0\rangle\,. \tag{E.4}$$

Here ϵ is a polarization tensor and in complex coordinate notation $\epsilon\cdot b = \epsilon_1 b$ and similarly $p \cdot b = p_1 b$.

We have written only the part of the graviton vertex which is most relevant for the $N \to \infty$ limit. Other parts will be suppressed by additional powers of $p_1 \leq p$, the graviton energy being

$$p = \frac{l}{2\sqrt{N}}\,.$$

The computation of the matrix element gives:

$$A_R = \epsilon_1 \frac{(-\sqrt{2}p_1)^{l-1}}{(l-1)!}\sqrt{\frac{N!}{(N-l)!}}\,. \tag{E.5}$$

and similarly for A_L with $\epsilon_1, p_1 \to \epsilon_2, p_2$.

In the limit $N \to \infty$ at fixed l, $A_{R,L} \to c(l,\theta)\ \sqrt{N}$ where $c(l,\theta)$ depends on the angle of the graviton momentum with the (complex) directions $1, 2$ respectively.

We thus get the decay rate for the present channel:

$$D(N,l) = \frac{1}{\sqrt{N}} g_s^2 \int d\Omega_9\ |A_R\cdot A_L|^2 \frac{p^7}{\sqrt{N}}\,, \tag{E.6}$$

where the last phase space factor comes from

$$\int \frac{p^8 dp}{p\sqrt{p^2+N-l}}\delta(\sqrt{N} - p - \sqrt{p^2+N-l}\ ) = \frac{p^7}{\sqrt{N}}\,,$$

and we have ignored some numerical factors. The integration over the solid angle gives a suppression factor which depends on l but not on N.

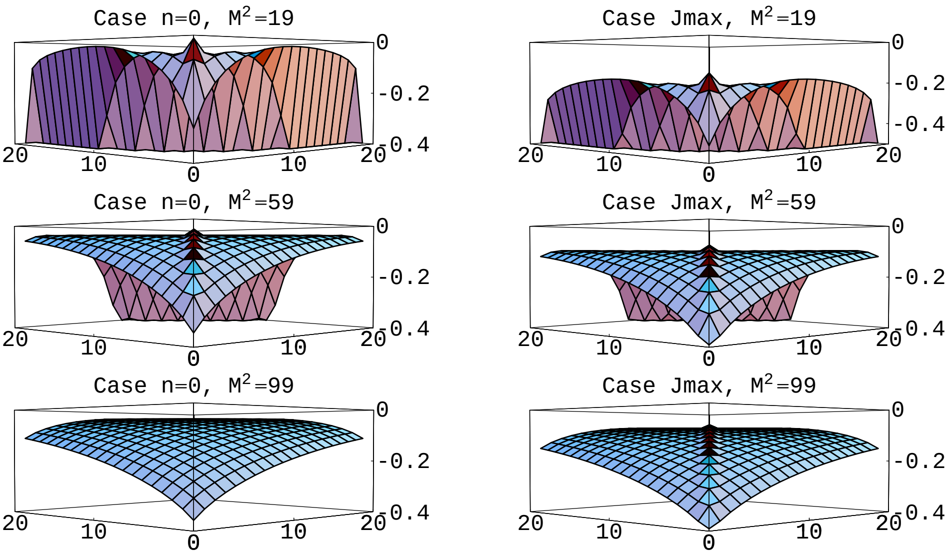

Figure 5: Three-dimensional plot of S_0.

Finally we get the behavior:

$$D(N,l) \cong g_s^2\; c_0(l)\; N^{-5/2}\,, \qquad N \gg 1\,. \tag{E.7}$$

We think that other channels are also relevant in the process $M \to M_1 + massless$. In those channels the massive final state can be different from the class of "$n = k$" states considered here.

F. Three-dimensional plot of the logarithm of the decay rate for $n = 0$ and $J_{\max}$ cases

Here we show graphics representing the shape of the logarithm of the decay rate (divided by $2M^2$), called S_0 in the text, as a function of M_1^2, M_2^2.

The figures 5 have been obtained focusing on an interval in the phase space in which $0 \leq M_{1,2}^2 \leq 20$, and limiting the value of S_0 to a range $[-0.4, 0]$.

We have chosen a perspective so as to have a clear view of the distance between the maxima and the plane $S_0 = 0$ (cf. also figures 2 and 3).

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